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An examination of the edge effect in a cantilever beam

R. H. BRYANT

Department of Materials Engineering, University of Illinois at Chicago Circle, Chicago, Illinois 60680, *USA*

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SUMMARY
A solution of the plane elasticity problem for the cantilever beam is presented. The classical solution in the interior region of the beam is developed with the aid of a geometric parameter. The effect of local transverse deformations and distortions are accounted for in the solution near the clamped edge. Some numerical results are presented.

1. Introduction

The classical plane stress solution of the cantilever beam is well known [1]. The solution, obtained by the use of an Airy stress function, is satisfied at the support by requiring that either the slope of the deformed cross-section of the beam is set equal to zero at the beams centerline, or the slope of the centerline itself is set to zero. The distorted cross-section is then without the requisite zero displacements, throughout its depth, needed to satisfy a truly fixed edge.

It is the purpose of this paper to investigate the solution of the cantilever beam problem and especially the edge solution near the support. Associated problems have been investigated by others, Benthem [4], Torvik [3] and Gusein–Zane [7] to name only three. In these papers the mathematical formulation and/or the method of solution were different from the problem presented here.

The method presented separates the solution into two parts: the "interior solution" and an "edge solution". This is accomplished by the use of parametric expansions [2], where two locally valid expansions of the stresses and displacements, in terms of a geometric parameter ε, is accomplished. The substitution of these expansions into the elasticity equations yield a sequence of systems of differential equations for the determination of the expansion coefficients. The solutions in the two regions are then matched and all the boundary conditions are satisfied. The final result is accomplished by means of the principle of virtual complementary work [5]. In accomplishing the solution the effect of enforcing elementary bending theory and the distortion of the beams cross-section are treated separately. The superposition of the solutions would, of course, yield the final result.

2. Formulation of exact equations

Consider a rectangular strip of height $2H$, unit thickness and length L made from an isotropic, homogeneous, elastic material (see Fig. 1). It is assumed that the stress and displacement description of such a strip can be obtained by the use of the plane stress elasticity equations. The end $X=0$, $-H \leqq Y \leqq +H$ is taken to be fixed, while the end $X=L$, $-H \leqq Y \leqq +H$ is subjected to prescribed stresses. The surfaces $Y=\pm H$, $0 \leqq X \leqq L$ are prescribed to be free of surface tractions. We further assume that the length of the strip is much larger than the height.

Define a parameter ε and dimensionless coordinates by

$$\varepsilon = H/L\,, \quad y = Y/H\,, \quad x = X/L \tag{1}$$

where in accordance with our assumptions $\varepsilon \ll 1$.

The displacements in dimensionless form are defined by

$$u = u_1 \frac{\sigma}{E} L, \quad v = u_2 \frac{\sigma}{E} L. \tag{2}$$

The dimensionless stresses are defined as

$$\sigma_{xx} = \sigma t_{xx}, \quad \sigma_{yy} = \sigma t_{yy}, \quad \sigma_{xy} = \sigma t_{xy}. \tag{3}$$

In the above σ is a representative stress level and E is Young's modulus.

In terms of the defined variables (1), (2) and (3) the equilibrium equations can be expressed as

$$\varepsilon \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} = 0, \quad \varepsilon \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0 \tag{4}$$

while the constitutive relations become

$$\frac{\partial u_1}{\partial x} = t_{xx} - \nu t_{yy}, \quad \frac{\partial u_2}{\partial y} = \varepsilon (t_{yy} - t_{xx}), \quad \frac{\partial u_1}{\partial y} = -\varepsilon \frac{\partial u_2}{\partial x} + 2\varepsilon(1+2\nu) t_{xy}. \tag{5}$$

The associated boundary conditions include the displacement conditions on the clamped end,

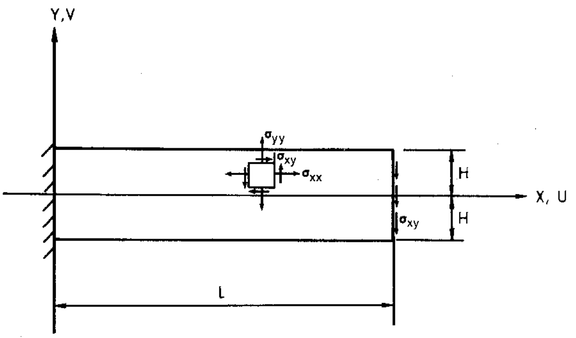

Figure 1. The cantilever beam.

the stress conditions on the free end and the boundary $y = \pm 1$, namely

$$u_1(0, y) = u_2(0, y) = 0 \quad t_{xx}(1, y) = 0, \quad t_{xy}(1, y) = f(y, \varepsilon) \tag{6}$$
$$t_{yy}(x, \pm 1) = t_{xy}(x, \pm 1) = 0.$$

3. Formulation of the interior problem

We assume the displacements u_1 and u_2 can be asymptotically represented by a power series in terms of a sequence of "interior displacement coefficients" $u_1^{(n)}(x, y)$ and $u_2^{(n)}(x, y)$, in the following manner

$$\begin{aligned} u_1(x, y;\ \varepsilon) &= u_1^{(1)}\varepsilon + u_1^{(3)}\varepsilon^3 + \ldots \\ u_2(x, y;\ \varepsilon) &= u_2^{(0)} + u_2^{(2)}\varepsilon_2 + \ldots \end{aligned} \tag{7}$$

It follows from Eq. (5) that each stress component may be represented by a power series in ε in terms of a sequence of "interior stress coefficients" $t_{xx}^{(n)}(x, y)$, $t_{xy}^{(n)}(x, y)$ and $t_{yy}^{(n)}(x, y)$ in the following way

$$\begin{aligned} t_{xx}(x, y;\ \varepsilon) &= t_{xx}^{(1)}\varepsilon + t_{xx}^{(3)}\varepsilon^3 + \ldots \\ t_{xy}(x, y;\ \varepsilon) &= t_{xy}^{(2)}\varepsilon^2 + t_{xy}^{(4)}\varepsilon^4 + \ldots \\ t_{yy}(x, y;\ \varepsilon) &= t_{yy}^{(3)}\varepsilon^7 + t_{yy}^{(5)}\varepsilon^5 + \ldots \end{aligned} \tag{8}$$

Substituting (7) and (8) into (5) and (4), we obtain

$$\frac{\partial}{\partial x}(u_1^{(1)}\varepsilon+u_1^{(3)}\varepsilon^3+\dots) = (t_{xx}^{(1)}\varepsilon+t_{xx}^{(3)}\varepsilon^3+\dots)-\nu(t_{yy}^{(3)}\varepsilon^3+t_{yy}^{(5)}\varepsilon^5+\dots) \tag{9}$$

$$\frac{\partial}{\partial y}(u_2^{(0)}+u_2^{(2)}\varepsilon^2+\dots) = (t_{yy}^{(3)}\varepsilon^3+t_{yy}^{(5)}\varepsilon^5+\dots)-\nu(t_{xx}^{(1)}\varepsilon+t_{xx}^{(3)}\varepsilon^3+\dots) \tag{10}$$

$$\frac{\partial}{\partial y}(u_1^{(1)}\varepsilon+u_1^{(3)}\varepsilon^3+\dots) = -\frac{\partial}{\partial x}(u_2^{(0)}\varepsilon+u_2^{(2)}\varepsilon^3+\dots)+2(1+2\nu)(t_{xy}^{(2)}\varepsilon^3+t_{xy}^{(4)}\varepsilon^5+\dots) \tag{11}$$

and

$$\frac{\partial}{\partial x}(t_{xx}^{(1)}\varepsilon^2+t_{xx}^{(3)}\varepsilon^4+\dots)+\frac{\partial}{\partial y}(t_{xy}^{(2)}\varepsilon^2+t_{xy}^{(4)}\varepsilon^4+\dots)=0 \tag{12}$$

$$\frac{\partial}{\partial x}(t_{xy}^{(2)}\varepsilon^3+t_{xy}^{(4)}\varepsilon^5+\dots)+\frac{\partial}{\partial y}(t_{yy}^{(3)}\varepsilon^3+t_{yy}^{(5)}\varepsilon^5+\dots)=0 \tag{13}$$

respectively. The tractions at $x=1$ are taken as

$$t_{xx}(1, y;\ \varepsilon)=0\,, \qquad t_{xy}(1, y;\ \varepsilon)=-\tfrac{3}{2}(1-y^2)\varepsilon^2\,. \tag{14}$$

The first set of equations obtained from (9–13) are

$$\frac{\partial u_1^{(1)}}{\partial x}=t_{xx}^{(1)} \quad \frac{\partial u_2^{(0)}}{\partial y}=0 \quad \frac{\partial u_1^{(1)}}{\partial y}=-\frac{\partial u_2^{(0)}}{\partial x}$$

$$\frac{\partial t_{xy}^{(2)}}{\partial y}=-\frac{\partial t_{xx}^{(1)}}{\partial x} \quad \frac{\partial t_{yy}^{(3)}}{\partial y}=-\frac{\partial t_{xy}^{(2)}}{\partial x} \tag{15}$$

Integration of (15) yields

$$u_2^{(0)}=U_2^{(0)}(x)\,, \quad u_1^{(1)}=U_1^{(1)}(y)-y\frac{dU_2^{(0)}}{dx}\,,$$

$$t_{xx}^{(1)}=\frac{dU_1^{(1)}}{dx}-y\frac{d^2U_2^{(0)}}{dx^2}\,, \quad t_{xy}^{(2)}=T_{xy}^{(2)}(x)-y\frac{d^2U_1^{(1)}}{dx^2}-\frac{y^2}{2}\frac{d^3U_2^{(0)}}{dx^3} \tag{16}$$

$$t_{yy}^{(3)}=T_{yy}^{(3)}(x)-y\frac{dT_{xy}^{(2)}}{dx}+\frac{y^2}{2}\frac{d^3U_1^{(1)}}{dx^3}-\frac{y^3}{6}\frac{d^4U_2^{(0)}}{dx^4}$$

where $U_2^{(0)}$ and $U_1^{(1)}$ are displacements defined at $y=0$, and $T_{xy}^{(2)}$ and $T_{yy}^{(3)}$ are the corresponding stress components. The conditions

$$t_{xy}^{(2)}(x, \pm 1)=0\,, \quad t_{yy}^{(3)}(x, \pm 1)=0 \tag{17}$$

yields

$$T_{xy}^{(2)}=-\tfrac{1}{2}\frac{d^3U_2^{(0)}}{dx^3}\,, \quad \frac{d^2U_1^{(1)}}{dx^2}=0\,, \quad T_{yy}^{(3)}=0\,. \tag{18}$$

From (18) we obtain

$$U_1^{(1)}=A_1x+A_2\,. \tag{19}$$

Here A_1 and A_2 are constants.

From Eqs. (14) and (11) we have

$$-\tfrac{1}{2}(1-y^2)\frac{d^3U_2^{(0)}}{dx^3}=-\tfrac{3}{2}(1-y^2) \tag{20}$$

or

$$U_2^{(0)}=\frac{x^3}{2}+A_3x^2+A_4x+A_5 \tag{21}$$

where A_3, A_4 and A_5 are constants.

Equations (14), (19) and (21) yields

$$A_1 = 0\,, \quad A_4 = -\tfrac{3}{2}\,, \quad t_{xx}^{(1)} = -3y(x-1)\,. \tag{22}$$

The final expression for the first approximation displacement and stress components are given by

$$\begin{aligned} &u_1^{(1)} = -\tfrac{3}{2}(x^2-2x)-A_4y+A_2\,, \quad u_2^{(0)} = \frac{x^3}{2} - \tfrac{3}{2}x^2+A_4x+A_5 \\ &t_{xx}^{(1)} = -3y(x-1)\,, \quad t_{xy}^{(2)} = -\tfrac{3}{2}(1-y^2)\,, \quad t_{yy}^{(3)} = 0\,. \end{aligned} \tag{23}$$

For a second approximation theory, (9–13) yield

$$\begin{aligned} &u_1^{(3)} = \frac{\nu y^3}{3} - A_6y+A_7\,, \quad u_2^{(2)} = -\nu y^2(x-1)+A_6x+A_8 \\ &t_{xx}^{(3)} = t_{xy}^{(4)} = t_{yy}^{(5)} = 0\,. \end{aligned} \tag{24}$$

Again the subscripted constants denoted by A are rigid body displacements.

Eqs. (23) and (24) represent the classical solutions for the strip problem (i.e. beam theory and plane elasticity). An investigation of higher approximations indicates that no further corrections to the first and second approximation is obtainable.

4. Formulation of the edge problem

The solutions obtained in the previous section will, in general, be independent of the boundary conditions at the fixed edge. Therefore the interior solution will hold in some region away from the edge. Near the edge, i.e. in the "boundary layer", the solution must change rapidly from that of the interior solution so as to be able to satisfy the edge conditions of the exact theory.

To obtain an expression that uniformly represents the solution up to and including the edge, we employ a "stretching" of the independent variable x

$$\xi = \frac{x}{\varepsilon} = \frac{X}{H}\,. \tag{25}$$

Transforming the independent variable, we obtain

$$u(x, y;\ \varepsilon) = u(\xi, y;\ \varepsilon)\,, \quad t(x, y;\ \varepsilon) = t(\xi, y;\ \varepsilon) \tag{26}$$

Here u and t are generic symbols representing the displacements and stresses. We assume that u and t can be represented by a power series in ε in terms of a sequence of "boundary layer displacement and stress coefficients", written in terms of generic symbols, $\mu^{(n)}(\xi, y)$, $\tau^{(n)}(\xi, y)$ where

$$\begin{aligned} &u(\xi, y; \varepsilon) = \sum_{n=0}^{N} \mu^{(n)}(\xi, y)\varepsilon^n\,, \quad \mu^{(n)} = 0 \text{ if } n<0 \\ &t(\xi, y; \varepsilon) = \sum_{n=0}^{N} \tau^{(n)}(\xi, y)\varepsilon^n\,, \quad \tau^{(n)} = 0 \text{ if } n<0\,. \end{aligned} \tag{27}$$

Introducing (27) into (5) and (4) yields the constitutive equations

$$\frac{\partial}{\partial \xi}(\mu_1^{(0)}+\mu_1^{(1)}\varepsilon+\ldots) = \varepsilon(\tau_{\xi\xi}^{(0)}+\tau_{\xi\xi}^{(1)}\varepsilon+\ldots)-\nu\varepsilon(\tau_{yy}^{(0)}+\tau_{yy}^{(1)}\varepsilon+\ldots) \tag{28}$$

$$\frac{\partial}{\partial y}(\mu_2^{(0)}+\mu_2^{(1)}\varepsilon+\ldots) = \varepsilon(\tau_{yy}^{(0)}+\tau_{yy}^{(1)}\varepsilon+\ldots)-\nu\varepsilon(\tau_{\xi\xi}^{(0)}+\tau_{\xi\xi}^{(1)}\varepsilon+\ldots) \tag{29}$$

$$\frac{\partial}{\partial y}(\mu_1^{(0)}+\mu_1^{(1)}\varepsilon+\ldots) = -\frac{\partial}{\partial \xi}(\mu_2^{(0)}+\mu_2^{(1)}\varepsilon+\ldots)+2(1+\nu)\varepsilon(\tau_{\xi y}^{(0)}+\tau_{\xi y}^{(1)}\varepsilon+\ldots) \tag{30}$$

and the equilibrium equations

$$\frac{\partial}{\partial \xi}(\tau_{\xi\xi}^{(0)}+\tau_{\xi\xi}^{(1)}\varepsilon+\ldots)+\frac{\partial}{\partial y}(\tau_{\xi y}^{(0)}+\tau_{\xi y}^{(1)}\varepsilon+\ldots)=0 \tag{31}$$

$$\frac{\partial}{\partial \xi}(\tau_{\xi y}^{(0)}+\tau_{\xi y}^{(1)}\varepsilon+\ldots)+\frac{\partial}{\partial y}(\tau_{yy}^{(0)}+\tau_{yy}^{(1)}\varepsilon+\ldots)=0\,. \tag{32}$$

Setting the same powers of ε to zero in (28–32) yields

$$\begin{aligned}
\tau_{\xi\xi}^{(n-1)} &= \frac{1}{1-\nu^2}\left(\frac{\partial \mu_1^{(n)}}{\partial \xi}+\nu\frac{\partial u_2^{(n)}}{\partial y}\right)\\
\tau_{yy}^{(n-1)} &= \frac{1}{1-\nu^2}\left(\frac{\partial \mu_2^{(n)}}{\partial y}+\nu\frac{\partial \mu_1^{(n)}}{\partial \xi}\right)\\
\tau_{\xi y}^{(n-1)} &= \frac{1}{2(1+\nu)}\left(\frac{\partial \mu_1^{(n)}}{\partial y}+\frac{\partial \mu_2^{(n)}}{\partial \xi}\right)
\end{aligned} \tag{33}$$

and

$$\frac{\partial \tau_{\xi\xi}^{(n-1)}}{\partial \xi}+\frac{\partial \tau_{\xi y}^{(n-1)}}{\partial y}=0\,,\qquad \frac{\partial \tau_{\xi y}^{(n-1)}}{\partial \xi}+\frac{\partial \tau_{yy}^{(n-1)}}{\partial y}=0\,. \tag{34}$$

For $n=0$ these equations yield

$$\mu_1^{(0)}=B_1 y\,,\qquad \mu_2^{(0)}=-B_1 \tag{35}$$

where B_1 is an arbitrary constant to be determined. For $n \geqq 1$ substitution of (33) into (34) yields

$$\frac{\partial^2 \mu_1^{(n)}}{\partial \xi^2}+\frac{(1+\nu)}{2}\frac{\partial^2 \mu_2^{(n)}}{\partial \xi \partial y}+\frac{(1-\nu)}{2}\frac{\partial^2 \mu_1^{(n)}}{\partial y^2}=0 \tag{36}$$

$$\frac{\partial^2 \mu_2^{(n)}}{\partial y^2}+\frac{(1+\nu)}{2}\frac{\partial^2 \mu_1^{(n)}}{\partial \xi \partial y}+\frac{(1-\nu)}{2}\frac{\partial^2 \mu_2^{(n)}}{\partial \xi^2}=0\,. \tag{37}$$

The associated boundary conditions require that the surface $y=\pm 1$ are free of tractions and the displacements at $\xi=0$ is specified. Consider solutions to (36) and (37) of the form

$$\mu^{(n)}(\xi, y)=a^{(n)}v^{(n)}(y,\beta)\exp[-\beta^{(n)}\xi]\,. \tag{38}$$

Here $a^{(n)}$ are constants associated with $\beta^{(n)}$. Substituting (38) into (36) and (37) and accounting for the boundary conditions at $y=\pm 1$ results in the following transcendental relationship

$$\sin 2\beta_k = -2\beta_k \tag{39}$$

$$\sin 2\beta_k = 2\beta_k \tag{40}$$

The roots of Eqs. (39) and (40) with positive real parts occur in pairs, β_k and $\bar{\beta}_k$ (its conjugate). Equation (38) may be then taken in the form

$$\mu^{(n)}(\xi, y)=\mathrm{Re}\sum_{k=1}^{2k}{}_k a^{(n)}{}_k v^{(n)}(y,\beta_k^{(n)})\exp[-\beta_k^{(n)}\xi] \tag{41}$$

where Re denotes the real part. For non-trivial solutions of (39) we have displacements which are symmetrical in y. The displacements and the corresponding stresses are given by

$$
\begin{aligned}
{}_k v_1 &= -\left[\frac{2}{\beta_k(1+\nu)} + \tan\beta_k\right]\cos\beta_k y + y\sin\beta_k y \\
{}_k v_2 &= \left[\frac{1-\nu}{\beta_k(1+\nu)} - \tan\beta_k\right]\cos\beta_k y - y\cos\beta_k y \\
{}_k\tau_{\xi\xi} &= \frac{1}{(1+\nu)}\left\{\left[\frac{2}{(1+\nu)} + \beta_k\tan\beta_k\right]\cos\beta_k y - \beta_k y\sin\beta_k y\right\}. \\
{}_k\tau_{\xi y} &= \frac{1}{(1+\nu)}\left[(1+\beta_k\tan\beta_k)\sin\beta_k y - \beta_k y\cos\beta_k y\right] \\
{}_k\tau_{yy} &= \frac{1}{(1+\nu)}\quad - \quad \frac{2}{(1+\nu)} + \beta_k\tan\beta_k \quad \cos\beta_k y + \beta_k y\sin\beta_k y\Big\}
\end{aligned}
\tag{42}
$$

where the superscript n has been omitted. The antisymmetric displacements correspond to non-trivial solutions of (40) and are given by

$$
\begin{aligned}
{}_k v_1 &= \left[\frac{(1-\nu)}{\beta_k(1+\nu)} + \tan\beta_k\right]\sin\beta_k y + y\cos\beta_k y \\
{}_k v_2 &= \left[\frac{2}{\beta_k(1+\nu)} - \tan\beta_k\right]\cos\beta_k y + y\sin\beta_k y \\
{}_k\tau_{\xi\xi} &= \frac{1}{(1+\nu)}\left[(1+\beta_k\tan\beta_k)\sin\beta_k y + \beta_k y\cos\beta_k y\right] \\
{}_k\tau_{\xi y} &= \frac{1}{(1+\nu)}\left[\beta_k\tan\beta_k\cos\beta_k y - \beta_k y\sin\beta_k y\right] \\
{}_k\tau_{yy} &= -\frac{1}{(1+\nu)}\left[(1-\beta_k\tan\beta_k)\sin\beta_k y - \beta_k y\cos\beta_k y\right].
\end{aligned}
\tag{43}
$$

A straightforward calculation will show that the stress resultants

$$
\int_{-1}^{1} {}_k\tau_{\xi\xi}\,dy\,, \qquad \int_{-1}^{1} y_k\tau_{\xi\xi}\,dy\,, \qquad \int_{-1}^{1} {}_k\tau_{\xi y}\,dy\,,
$$

are identically equal to zero. Therefore the solutions given by (42) and (43) are self-equilibrating and do not provide equilibrium conditions for the strip. In addition, the solutions in the boundary layer should match the interior solution for large ξ. To fulfill the above requirements we will add to solutions (42) and (43) the interior solution for ε approaching zero. Specifically, we assume that each $u^{(n)}(x, y)$, $t^{(n)}(x, y)$ near $x=0$ has a Taylor series expansion

$$
u^{(n)}(x, y) = \sum_{m=0}^{\infty} f_m^{(n)}(x, y)\,; \qquad t^{(n)}(x, y) = \sum_{m=0}^{\infty} f_m^{(n)}(x, y) \tag{44}
$$

where the generic symbol $f_m^{(n)}(x, y)$ is

$$
f_m^{(n)}(x, y) = \begin{cases} \dfrac{1}{m!}\dfrac{\partial^m u^{(n)}(x, y)}{\partial x^m}\Big|_{x=0}\,; \quad \dfrac{1}{m!}\dfrac{\partial^m t^{(n)}(x, y)}{\partial x^m}\Big|_{x=0} & (n \geqq 0) \\ 0 & (n < 0) \end{cases} \tag{45}
$$

From (7–8) and (25) we have in the neighborhood of $x=0$

$$
\begin{aligned}
u(x, y;\ \varepsilon) &= \sum_{n=0}^{N} \sum_{m=0}^{n} f_m^{(n)}(\xi\varepsilon, y)\varepsilon^n = \sum_{n=0}^{N} \sum_{m=0}^{n} f_m^{(n-n)}(0, y)\xi^m \varepsilon^n \\
t(x, y;\ \varepsilon) &= \sum_{n=0}^{N} \sum_{m=0}^{n} f_m^{(n-m)}(0, y)\xi^m \varepsilon^n .
\end{aligned}
\tag{46}
$$

Equation (46) represents the interior solutions in terms of ξ and y. The edge solution is finally determined from (45) and (46):

$$
\begin{aligned}
\hat{\mu}^{(n)}(\xi, y) &= \mu^{(n)}(\xi, y) + \sum_{m=0}^{n} u^{(n-m)}(0, y)\xi^m \\
\hat{\tau}^{(n)}(\xi, y) &= \tau^{(n)}(\xi, y) + \sum_{m=0}^{n} t^{(n-m)}(0, y)\xi^m
\end{aligned}
\tag{47}
$$

for each n.

For the cantilever beam problem, if $n=0$ the edge problem is represented by Eqns. (23), (41), and (46) as

$$
\hat{\mu}_1^{(0)} = B_1 y , \qquad \hat{\mu}_2^{(0)} = -B_1 \xi + A_5 . \tag{48}
$$

The boundary conditions at $\xi=0$ require $B_1 = A_5 = 0$. Considering $n=1$ next, we have from (23), (41) and (47)

$$
\begin{aligned}
&\hat{\mu}_1^{(1)} = \mu_1^{(1)} - A_4 y + A_2 , \qquad \hat{\mu}_2^{(1)} = \mu_2^{(1)} = \mu_2^{(1)} + A_4 \xi \\
&\hat{\tau}_{\xi\xi}^{(0)} = \tau_{\xi\xi}^{(0)} , \qquad \hat{\tau}_{\xi y}^{(1)} = \tau_{\xi y} , \qquad \hat{\tau}_{yy}^{(0)} = \tau_{yy}^{(0)} .
\end{aligned}
\tag{49}
$$

If $A_2 = A_4 = 0$ is assumed, the trivial solution must be taken as the result. For $n=2$, the first order edge problem is obtained from (23), (24), (41), and (47) which yield

$$
\begin{aligned}
\hat{\mu}_1^{(2)} &= \text{Re} \sum_{k=1}^{2K} {}_k a_k v_1 \, \text{e}^{-\beta_k \xi} + {}_{2K+1} a \, \tfrac{3}{2} \xi y \\
\hat{\mu}_2^{(2)} &= \text{Re} \sum_{k=1}^{2K} {}_k a_k v_2 \, \text{e}^{-\beta_k \xi} + {}_{2K+1} a \left(\frac{\nu}{2} y^2 - \xi^2 \right) + A_8 \\
\hat{\tau}_{\xi\xi}^{(1)} &= \text{Re} \sum_{k=1}^{2K} {}_k a_k \tau_{\xi\xi} \text{e}^{-\beta_k \xi} + {}_{2K+1} a \, 3y \\
\hat{\tau}_{\xi y}^{(1)} &= \text{Re} \sum_{k=1}^{2K} {}_k a_k \tau_{\xi y} \text{e}^{-\beta_k \xi} \\
\hat{\tau}_{yy}^{(1)} &= \text{Re} \sum_{k=1}^{2K} {}_k a_k \tau_{yy} \text{e}^{-\beta_k \xi} .
\end{aligned}
\tag{50}
$$

In (50) only the first K values for the eigenfunctions which occur in pairs are considered. The constant ${}_{2K+1}a$ represents the coefficients of the deformation and corresponding stress which result from the interior solution. The reason for its introduction will become apparent in the next section. All the constants ${}_k a$ are to be normalized by Re$({}_{2K+1}a)$ to obtain the final solution. It should be noted that only the antisymmetric solution obtained from (43) are included in (50). The constant A_5 may be set equal to zero without effecting the results.

Finally the second approximation is obtained from (23), (24), (41) and (47) for $n=3$. The result is

$$\hat{\mu}_1^{(3)} = \text{Re} \sum_{k=1}^{2K} {}_k a_k v_1 \, \mathrm{e}^{-\beta_k \xi} + {}_{2K+1}a \left(\frac{v}{6} y^3 - \xi^2 y \right) - A_9 y + A_7$$

$$\hat{\mu}_2^{(3)} = \text{Re} \sum_{k=1}^{2K} {}_k a_k v_2 \, \mathrm{e}^{-\beta_k \xi} + {}_{2K+1}a \left(\frac{v}{2} y^2 - \tfrac{1}{3} \xi^3 \right)$$

$$\hat{\tau}_{\xi\xi}^{(2)} = \text{Re} \sum_{k=1}^{2K} {}_k a_k \tau_{\xi\xi} \mathrm{e}^{-\beta_k \xi} - {}_{2K+1} 3a \xi y \tag{51}$$

$$\hat{\tau}_{\xi y}^{(2)} = \text{Re} \sum_{k=1}^{2K} {}_k a_k \tau_{\xi y} \mathrm{e}^{-\beta_k \xi} - {}_{2K+1}a \, \tfrac{3}{2}(1-y^2)$$

$$\hat{\tau}_{yy}^{(2)} = \text{Re} \sum_{k=1}^{2K} {}_k a_k \tau_{yy} \mathrm{e}^{-\beta_k \xi} .$$

The terms in (51) are defined as in (50). No generality is lost by taking $A_7 = A_9 = 0$.

The effect of the deformations parallel to the fixed edge are seen to occur in (50) while the distortion of the cross-section of the beam occurs in (51).

5. Problem solution

Equations (50) and (51) will be solved as two separate systems. The results need only be superimposed to yield the total solution. To determine the constants ${}_k a$ the principle of virtual complementary work is used in the form

$$\int_v (\varepsilon_{\xi\xi} \delta\hat{\tau}_{\xi\xi} + \varepsilon_{\xi y} \delta\hat{\tau}_{\xi y} + \varepsilon_{yy} \delta\hat{\tau}_{yy}) dV = \int_{S_1} (U_1 \delta\hat{\tau}_{\xi\xi} + U_2 \delta\hat{\tau}_{\xi y}) dy + $$

$$+ \int_{S_2} (U_1 \delta t_{\xi\xi} + U_2 \delta t_{\xi y}) dy \tag{52}$$

where $\varepsilon_{\xi\xi}$, $\varepsilon_{\xi y}$, ε_{yy} are the strains, U_1, U_2 are prescribed displacements, $\delta\hat{\tau}_{\xi\xi}$, $\delta\hat{\tau}_{\xi y}$, $\delta\hat{\tau}_{yy}$ the statically admissible stresses; $\delta\tau_{\xi\xi}$, $\delta\tau_{\xi y}$ the corresponding tractions for large ξ. S_1 and S_2 are the boundaries at $\xi = 0$ and ξ large respectively. The notation $\delta\hat{\tau} = \delta({}_k a)_k \hat{\tau}$ is used. It should be clear that U_1 and U_2 at S_2 are the displacements for the interior solution.

Applying the divergence theorem to (52), substituting (50) and (51) into the result and furthermore assuming that the decaying exponential terms are negligible for large ξ, the two integrals which occur at S_2 cancel each other. This is a consequence that the tractions and displacements from the edge solution match those of the interior solution on S_2. The result is

$$\sum_{m=1}^{2K+1} \text{Re} \left\{ \int_{S_1} \left[\sum_{k=1}^{2K+1} ({}_k a_k \hat{\mu}_{1m} \delta\bar{\hat{\tau}}_{\xi\xi} + {}_k a_k \hat{\mu}_2 \delta\bar{\hat{\tau}}_{\xi y}) - U_{1m} \delta\bar{\hat{\tau}}_{\xi\xi} + U_{2m} \delta\bar{\hat{\tau}}_{\xi y} \right] dy = 0 . \right. \tag{53}$$

In (53) the bar quantities indicate the complex conjugate of the stresses. The relation furnishes $2K+1$ equations for $2K+1$ unknowns when U_1 and U_2 are specified.

7. Numerical results

The roots of the transcendental equations were determined numerically with the aid of an electronic computer. The values of v and L/H are taken to be 0.25 and 20 respectively. The integrals in (53) were performed by hand and the constants ${}_k a$ determined numerically.

The solution for the first order approximation was obtained by substituting (50) into (53) and setting $U_1 = Ay$ and $U_2 = 0$. After the constants ${}_k a$ are determined the rigid body displacements $\mu_1 = -Ay$ and $\mu_2 = A\xi$ are superimposed to achieve the desired result. In addition it was found that a rigid body translation of 0.001705 A was necessary to finally orientate the strip. This value of the rigid body translation corresponds to 30 pairs of terms. Of particular interest are the results at the clamped edge. The convergence of the horizontal displacement to the desired value (prior to the addition of the rigid body motion) is compared in Fig. 2 for two different numbers of pairs of eigenvalues. The lack of agreement was 0.19 percent over the entire cross-section for 10 pairs of terms. There is no noticeable difference for 30 pairs. The results for 20 pairs of terms are not included for clarity. The convergence for u_2 is not quite as good for the same number of terms.

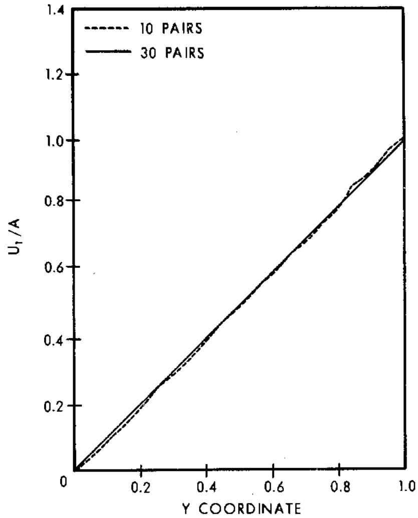

Figure 2. Horizontal displacement for the first approximation at $X=0$.

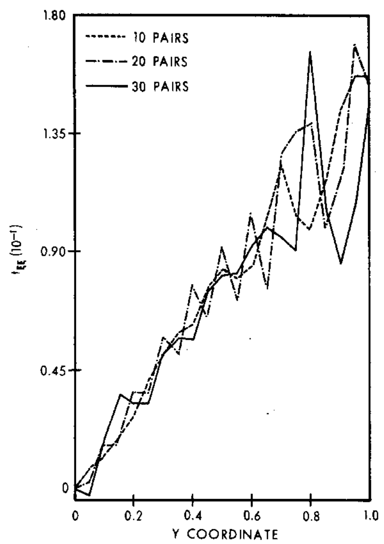

Figure 3. Normal stress for the first approximation at $X=0$.

The stress condition at the clamped edge is of particular interest because of the singularity which exists at the corners [6]. The normal and shearing stresses are shown in Figs. 3 and 4. The computed values are connected by straight lines for comparison with different numbers of eigenvalues. It is seen that the behavior is erratic over the entire plane of the cross-section. For the normal stresses the erratic behavior increases with a greater number of pairs of eigenvalues, especially away from the center of the edge toward the corner. This trend is not as apparent with the shear stresses. The stress t_{yy} is not plotted as its behavior is also erratic, and it was felt no new information would be added by including its results in the figure.

For the second approximation, the solution is obtained by substituting (51) into (53) and setting $U_1=0$, $U_2=A$. Only the rigid body translation $U_2=-A$ need be added to achieve the

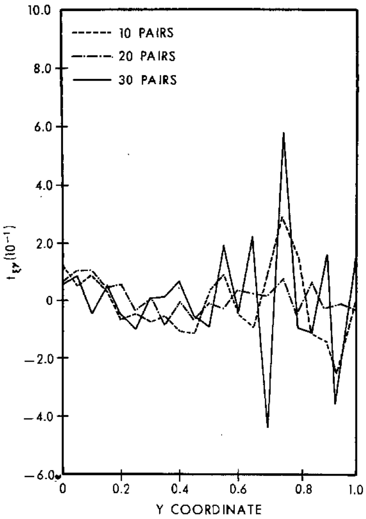

Figure 4. Shear stress for the first approximation at $X=0$.

desired result. The general behavior and quality of convergence are similar to those of the first approximation. That is, the agreement with displacements are good, but the stresses were again erratic in behavior for the number of terms in the computation. It should be mentioned that the normal stresses were about one order of magnitude less than in the first approximation.

The final result is obtained by superposition of the two solutions in accordance with (47). For both solutions the self-equilibrating terms are negligible at $X=0.4$ H.

8. Conclusions

Through the use of parameter expansions, the formulation of the cantilever beam problem was obtained in a systematic manner. The classical solution was obtained in the region away from the edge without reference to the end fixity. By use of a stretched coordinate, a self-equilibrating solution valid in the region adjacent to the edge was obtained. The classical solution was then superimposed on this solution in a consistent manner at each level of approximation. The unknown constants appearing in the edge solution were then determined by the principle of virtual complementary work. The analysis of similar problems in plane elasticity are possible by the same technique.

For the number of terms used in the computation, the series solution for the stresses at the edge did not converge. This was particularly noticeable away from the center of the beam axis. The trouble encountered here was noted in Torvik's paper [3] for a similar problem. It was noted in that paper that the singularity in the stress field at the corners may have well caused the lack of convergence.

The solution for the displacements at the edge converged quite well.

REFERENCES

[1] S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw Hill, New York (1970).
[2] M. W. Johnson and E. Reissner, Parameter Expansions for a Class of Boundary Value Problems of Partial Differential Equations, *J. Soc. Indust. Appl. Math.*, 8 (1960) 389–396.
[3] P. J. Torvik, The Elastic Strip with Prescribed End Displacement, *S. J. Appl. Mech.*, 38 (1971) 929–936.
[4] J. P. Benthem, A Laplace Transform Method for the Solution of Semi-Infinite and Finite Strip Problems in Stress Analysis, *Quart. Jour. Mech. and Appl. Math.*, XVI (1963) 413–429.
[5] Y. C. Fung, Foundation of Solid Mechanics, Prentice Hall, New York (1970).
[6] F. C. Karal and S. N. Samuel Karp, Stress Behaviour in the Neighborhood of Sharp Corners, *Geophysics XXIX*, 3 (1964) 360–369.
[7] M. I. Gusein–Zade, On the Conditions of Existence of Decaying Solutions of the Two Dimensional Problem of the Theory of Elasticity for a Semi-Infinite Strip, *PMM*, 29, 2 (1964) 447–453.